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# Magnetic monopoles and nonassociative deformations of quantum theory

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**Abstract.** We examine certain nonassociative deformations of quantum mechanics and gravity in three dimensions related to the dynamics of electrons in uniform distributions of magnetic charge. We describe a quantitative framework for nonassociative quantum mechanics in this setting, which exhibits new effects compared to ordinary quantum mechanics with sourceless magnetic fields, and the extent to which these theoretical consequences may be experimentally testable. We relate this theory to noncommutative Jordanian quantum mechanics, and show that its underlying algebra can be obtained as a contraction of the alternative algebra of octonions. The uncontracted octonion algebra conjecturally describes a nonassociative deformation of three-dimensional quantum gravity induced by magnetic monopoles, which we propose is realised by a non-geometric Kaluza-Klein monopole background in M-theory.

## 1. Prologue: Three-dimensional quantum gravity

Applications of noncommutative geometry in physics are often motivated as providing a suitable mathematical framework for describing the modifications of spacetime geometry at very short distance scales which are expected in a quantum theory of gravity; the length scale at which such effects become important is usually understood to be the Planck length  $\ell_P$ . However, there are relatively few precise quantitative connections between noncommutative geometry and models of quantum gravity. One notable exception is quantum gravity in three spacetime dimensions; that three-dimensional quantum gravity naturally implies a “fuzziness” to short distance spacetime structure was noted already in the early work of ’t Hooft [1] and others, see e.g. [2], who observed that the spectrum of position operators in this theory is discrete. In this paper we shall be primarily concerned with the precise realisation of these structures obtained by [3], who consider a Ponzano-Regge spin foam model of three-dimensional quantum gravity coupled to spinless matter fields. After integrating out the gravitational degrees of freedom in this model, they obtain an effective scalar field theory on a noncommutative spacetime described via the deformed phase space commutation relations

$$\begin{aligned} [x^i, x^j] &= i\ell_P \varepsilon^{ijk} x^k, \\ [x_i, p_j] &= i\sqrt{\hbar^2 - \ell_P^2 p^2} \delta_{ij} - i\ell_P \varepsilon_{ijk} p_k, \\ [p_i, p_j] &= 0. \end{aligned} \tag{1.1}$$



The commutation relations among position coordinates  $x^i$  specify a Lie algebra noncommutative spacetime; the relations apply to both Euclidean signature, wherein the pertinent Lie group is  $SU(2)$  and which is the main focus of this paper, and also in Minkowski signature wherein the pertinent Lie group is  $SO(1, 2)$ .

The commutation relations (1.1) can be understood in the following way. A peculiar feature of quantum gravity in three dimensions is that the momenta  $p_i$  of a particle are bounded from above, with the bound set by the inverse of the Planck length  $\ell_P$ . In particular, we can take the momentum space to be a sphere of radius  $\ell_P^{-1}$ . Then the commutators in (1.1) simply reflect the fact that position coordinates act as derivations on this sphere, as implied by the usual canonical commutation relations. The deformation of the latter in (1.1) makes the noncommutative algebra invariant under a  $\kappa$ -deformation of the Poincaré group in three dimensions, with  $\kappa = \ell_P^{-1}$ , such that the phase space commutation relations define a noncommutative but associative algebra. Setting  $\ell_P = 0$  corresponds to turning off the effects of quantum gravity and leaves an undeformed canonical phase space in (1.1).

Many interesting physical properties of three-dimensional quantum gravity can be inferred from the relations (1.1). For example, they imply a deformation of the usual addition law for Fourier wavenumbers, which in Minkowski signature leads to the modified dispersion relations

$$E^2 = \mathbf{p}^2 c^2 - \left( \frac{\sinh(\ell_P \hbar^{-1} m c^2)}{\ell_P \hbar^{-1}} \right)^2. \quad (1.2)$$

In the limit  $\ell_P \rightarrow 0$  wherein quantum gravity effects are neglected, this is just the usual relativistic dispersion law for a scalar particle in three spacetime dimensions. The deformation by  $\ell_P \neq 0$  is interpreted as the statement that doubly special relativity arises precisely in the low energy limit of three-dimensional quantum gravity [3].

The purpose of this paper is to describe a novel conjectural nonassociative deformation of the three-dimensional quantum gravity algebra (1.1) which is implied by certain magnetic dual analogues of the types of nonassociative spacetime geometries that are anticipated to arise in non-geometric string theory and M-theory, see e.g. [4–13]. For a while we shall therefore leave the present setting of three-dimensional quantum gravity and explain how nonassociative deformations of phase space naturally appear in a simpler elementary quantum mechanical setting of magnetic monopole distributions (Section 2). By attempting to reconcile this particular nonassociative deformation of quantum mechanics with the nonassociative algebras of observables proposed in the early days of quantum mechanics by Jordan, von Neumann, Wigner and others to study the mathematical and conceptual foundations of quantum theory, we shall arrive at a new framework that conjecturally describes magnetic monopoles in the spacetime of three-dimensional quantum gravity (Section 3). We conclude with a proposed realisation of such a system within the framework of M-theory (Section 4).

## 2. Magnetic monopoles and nonassociative quantum mechanics

We shall start by discussing the quantum mechanics of electrons propagating in sources of magnetic charge, focusing on the case of uniform monopole distributions which are the suitable analog systems dual to certain non-geometric string backgrounds. In this latter case we argue that the conventional framework of quantum theory necessitates a nonassociative deformation, and then demonstrate that such a model of nonassociative quantum mechanics is not only possible, but physically sensible and potentially testable.

### 2.1. Magnetic sources in quantum mechanics

Electrons propagating in a magnetic field  $\mathbf{B}(\mathbf{x})$  under the influence of an external potential  $V(\mathbf{x})$  in three dimensions classically obey the Lorentz force law

$$\dot{\mathbf{p}} = \frac{e}{2mc} (\mathbf{p} \times \mathbf{B} - \mathbf{B} \times \mathbf{p}) + \nabla V(\mathbf{x}) \quad (2.1)$$

where  $\mathbf{p} = m\dot{\mathbf{x}}$  is the kinematical momentum (as opposed to the canonical momentum), and we have written the right-hand side in a form that generalises to the quantum theory. The corresponding Hamiltonian on phase space is the sum of kinetic and potential energies:

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m} \mathbf{p}^2 + V(\mathbf{x}) . \quad (2.2)$$

Then the quantum mechanical Heisenberg equations of motion

$$-i\hbar \dot{\mathbf{p}} = [\mathbf{H}, \mathbf{p}] \quad , \quad -i\hbar \dot{\mathbf{x}} = [\mathbf{H}, \mathbf{x}] \quad (2.3)$$

are compatible with the Lorentz force law and the relation  $\dot{\mathbf{x}} = \mathbf{p}/m$  only with the phase space commutation relations

$$[x^i, x^j] = 0 \quad , \quad [x^i, p_j] = i\hbar \delta^i_j \quad , \quad [p_i, p_j] = \frac{i\hbar e}{c} \varepsilon_{ijk} B^k , \quad (2.4)$$

which deform the canonical commutators to a noncommutative momentum space. Note that here and below everything is formulated without any reference to a vector potential, so that these considerations apply as well to the cases where the magnetic field  $\mathbf{B}$  has sources.

Globally, the noncommutativity can be described in terms of the magnetic translation operators

$$U(\mathbf{a}) = \exp\left(\frac{i}{\hbar} \mathbf{a} \cdot \mathbf{p}\right) \quad (2.5)$$

which generate finite translations  $U(\mathbf{a}) \mathbf{x} U(\mathbf{a})^{-1} = \mathbf{x} + \mathbf{a}$  due to the second commutator in (2.4). A simple calculation shows that they do not commute [14]:

$$U(\mathbf{a}_1) U(\mathbf{a}_2) = e^{\frac{i e}{\hbar c} \Phi_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2)} U(\mathbf{a}_1 + \mathbf{a}_2) = e^{\frac{2i e}{\hbar c} \Phi_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2)} U(\mathbf{a}_2) U(\mathbf{a}_1) , \quad (2.6)$$

where

$$\Phi_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2) = \int_{\langle \mathbf{a}_1, \mathbf{a}_2 \rangle_{\mathbf{x}}} \mathbf{B} \cdot d\mathbf{S} \quad (2.7)$$

is the magnetic flux through the plane of the triangle  $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle_{\mathbf{x}}$  based at  $\mathbf{x} \in \mathbb{R}^3$  and spanned by the vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$ .

One can define a 3-bracket on momentum space as the combination of iterated commutators

$$[p_1, p_2, p_3] := [p_1, [p_2, p_3]] + [p_2, [p_3, p_1]] + [p_3, [p_1, p_2]] \quad (2.8)$$

that would vanish if the brackets (2.4) satisfied the Jacobi identity; it is called the ‘Jacobiator’. One easily computes in this case

$$[p_1, p_2, p_3] = \frac{\hbar^2 e}{c} \nabla \cdot \mathbf{B} , \quad (2.9)$$

which expresses a further deformation of canonical phase space to a nonassociative momentum space. Globally, one easily computes using (2.6) that the magnetic translation operators thus do not associate:

$$(U(\mathbf{a}_1)U(\mathbf{a}_2))U(\mathbf{a}_3) = e^{\frac{ie}{\hbar c}\Phi_3(\mathbf{x};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)}U(\mathbf{a}_1)(U(\mathbf{a}_2)U(\mathbf{a}_3)), \quad (2.10)$$

where  $\Phi_3(\mathbf{x};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)$  is the sum of the magnetic fluxes through  $\langle\mathbf{a}_2,\mathbf{a}_3\rangle_{\mathbf{x}+\mathbf{a}_1}$  and  $\langle\mathbf{a}_1,\mathbf{a}_2+\mathbf{a}_3\rangle_{\mathbf{x}}$ , minus the fluxes through  $\langle\mathbf{a}_1,\mathbf{a}_2\rangle_{\mathbf{x}}$  and  $\langle\mathbf{a}_1+\mathbf{a}_2,\mathbf{a}_3\rangle_{\mathbf{x}}$ . Together this gives the magnetic flux through the faces of the tetrahedron  $\langle\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\rangle_{\mathbf{x}}$  based at  $\mathbf{x}$  and spanned by the vectors  $\mathbf{a}_1$ ,  $\mathbf{a}_2$  and  $\mathbf{a}_3$ , which using the divergence theorem can be written as the magnetic charge

$$\Phi_3(\mathbf{x};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3) = \int_{\langle\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\rangle_{\mathbf{x}}} \nabla \cdot \mathbf{B} \, dV \quad (2.11)$$

enclosed by the tetrahedron. There are three generic situations that can now arise.

First, in Maxwell theory without magnetic sources, we have  $\nabla \cdot \mathbf{B} = 0$ . In this case all variables associate. The commutation relations (2.4) may then be represented on the Hilbert space  $L^2(\mathbb{R}^3)$  of wavefunctions  $\psi(\mathbf{x})$  in a Schrödinger polarization by the kinematical momentum operators  $\mathbf{p} = -i\hbar\nabla - \frac{e}{c}\mathbf{A}$ , where  $\mathbf{A}$  is a globally defined vector potential for the magnetic field:  $\mathbf{B} = \nabla \times \mathbf{A}$ . The vanishing of the magnetic charge,  $\Phi_3 = 0$ , implies that the phase factor  $\Phi_2$  defines a 2-cocycle of the translation group of  $\mathbb{R}^3$ , and the relations (2.6) reflect the fact that the magnetic translation operators only form a projective representation of this group; as usual, such projective actions on wavefunctions  $\psi(\mathbf{x})$  are well-defined in quantum mechanics.

Next, suppose that there are magnetic sources so that  $\nabla \cdot \mathbf{B} \neq 0$ . Then the magnetic translation operators generate a nonassociative algebra unless the magnetic charge obeys

$$\frac{e\Phi_3}{2\pi\hbar c} = \text{integer}. \quad (2.12)$$

In this case  $\nabla \cdot \mathbf{B}$  is necessarily a sum of delta-functions, representing a set of isolated point-like magnetic monopoles, or else the quantization condition (2.12) would be incompatible with continuous variations of the position vectors  $\mathbf{a}_i \in \mathbb{R}^3$ . This constraint is simply the celebrated Dirac quantization condition [15], as first pointed out in the present context by Jackiw [16]; with it, the magnetic translations may be represented by linear operators on a Hilbert space, which necessarily associate. At the infinitesimal level, the Jacobiator (2.9) is only non-vanishing at the loci of the magnetic sources. For isolated magnetic monopoles, one can excise their locations from position space and quantize the system in the standard way on the excised space: Classically the electrons never reach the monopole locations because of angular momentum conservation [8], while at the quantum level the wavefunctions vanish at the magnetic charge loci. Moreover, the corresponding local vector potentials  $\mathbf{A}$ , and the magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$ , are singular at the monopole locations which can be interpreted in the standard way in terms of the singularities of Dirac strings which emanate from the monopoles [15]; for example, a single Dirac monopole sources the magnetic field

$$\mathbf{B}(\mathbf{x}) = \frac{\Phi_3}{4\pi} \frac{\mathbf{x}}{|\mathbf{x}|^3}. \quad (2.13)$$

Geometrically, the wavefunctions  $\psi(\mathbf{x})$  can be treated as sections and the local monopole potential  $\mathbf{A}$  as a connection of a non-trivial  $U(1)$ -bundle over the excised space [17], whose degree is precisely the magnetic charge (2.12).

In this paper we are interested in the case of a homogeneous magnetic charge density, wherein  $\nabla \cdot \mathbf{B}$  is constant. In this case one cannot excise the support of the magnetic

charge distribution, or else one would be left with empty space, and one needs to deal directly with the nonassociativity of momentum coordinates and the magnetic translations. This is a manifestation of the type of ‘local non-geometry’ that arises in flux compactifications of string theory; it originates from the fact that the uniform distribution of magnetic charge can no longer be described by a connection on a  $U(1)$ -bundle over a non-contractible space, as now a vector potential  $\mathbf{A}$  does not even exist locally when  $\nabla \cdot \mathbf{B} \neq 0$  everywhere, but rather it induces a connection  $F_{ij} = \varepsilon_{ijk} B^k$  on a (trivial)  $U(1)$ -gerbe on  $\mathbb{R}^3$ . In particular, there is no relation between the kinematical and canonical momentum in this case. Moreover, the magnetic flux  $\Phi_2$  is a 2-cochain which no longer defines a (weak) projective representation; its coboundary is the magnetic charge  $\Phi_3$  which defines a 3-cocycle of the translation group. Nonassociativity that is induced by a 3-cocycle controlling the associativity of operators can be elegantly dealt with by working in a suitable braided monoidal category of representations [18]. The geometric meaning of such higher structures in quantization is described in [19, 20], where the sections of the trivial  $U(1)$ -gerbe on  $\mathbb{R}^3$  with connection given by the magnetic field  $\mathbf{B}(\mathbf{x}) = \frac{1}{3} \mathbf{x}$  are shown to correspond to a certain 2-Hilbert space of Hermitian matrix-valued one-forms on  $\mathbb{R}^3$  with bicovariantly constant matrix-valued functions on  $\mathbb{R}^3$  as morphisms.

Previous algebraic characterisations of the commutation relations (2.4) have discussed their realisation as a nonassociative Malcev algebra [21, 22], and more generally as an alternative (or Jordan) algebra [23]; we shall come back to these points in Section 3, and indicate why these realisations are somewhat subtle from the perspective of this paper. Classically, the Lorentz force equations (2.1) no longer seem to be integrable, as the conservation of angular momentum of the Dirac monopole background is lost [8], while at the quantum level it is clear that some formalism of *nonassociative quantum mechanics* is needed; in the rest of this section we describe such a framework.

## 2.2. Phase space quantum mechanics

Operators which act on a separable Hilbert space necessarily associate, by definition. Hence the standard operator-state techniques of quantum mechanics are inadequate to handle systems with a non-trivial 3-cocycle that obstructs associativity. This can be presumably dealt with by a suitable higher operator-state formalism adapted to the quantum 2-Hilbert space of [20], but such techniques have not yet been developed. We can, however, appeal to the phase space formulation of quantum mechanics, which was succinctly developed by Grönwold [24], Moyal [25] and others, following earlier work of Weyl [26] and Wigner [27]; see e.g. [28] for a pedagogical introduction. One of the premises of this formalism is to treat position and momentum on equal footing, which are usually otherwise treated asymmetrically when choosing e.g. a Schrödinger polarization. It is through this that the formalism of star products was originally introduced.

Let us recall the basic dictionary in the case where  $\mathbf{B} = \mathbf{0}$ . Operators in general become functions on phase space, while observables correspond to real functions. Traces of operators are given by integrating functions over phase space. States  $\psi$  become real phase space quasi-probability distribution (Wigner) functions which are defined by Fourier transformation of their position space representations as

$$W_\psi(\mathbf{x}, \mathbf{p}) = \int \frac{d\mathbf{y}}{(2\pi)^3} \langle \mathbf{x} + \frac{\hbar}{2} \mathbf{y} | \psi \rangle \langle \psi | \mathbf{x} - \frac{\hbar}{2} \mathbf{y} \rangle e^{-i\mathbf{y} \cdot \mathbf{p}}, \quad (2.14)$$

such that the expectation value of an operator (function) is computed as

$$\langle A \rangle_\psi = \int d\mathbf{x} d\mathbf{p} W_\psi(\mathbf{x}, \mathbf{p}) A(\mathbf{x}, \mathbf{p}). \quad (2.15)$$

The operator product, which captures the crucial noncommutativity of quantum mechanics that

leads to e.g. uncertainty relations, becomes the associative noncommutative Moyal star product

$$A(\mathbf{x}, \mathbf{p}) \star B(\mathbf{x}, \mathbf{p}) = A\left(\mathbf{x} - \frac{i\hbar}{2} \nabla_{\mathbf{p}}, \mathbf{p} + \frac{i\hbar}{2} \nabla_{\mathbf{x}}\right) B(\mathbf{x}, \mathbf{p}) \quad (2.16)$$

which is defined by replacing the arguments of a function  $A(\mathbf{x}, \mathbf{p})$  by their Bopp shifts and letting the resulting differential operator act on the function  $B(\mathbf{x}, \mathbf{p})$ ; this can be properly defined when  $A(\mathbf{x}, \mathbf{p})$  is a polynomial and then continuing to power series expansions.

Given a Hamiltonian  $H(\mathbf{x}, \mathbf{p})$  on phase space, dynamics of observables is governed by the Heisenberg-type time evolution equations

$$\frac{dA}{dt} = \frac{[A, H]_{\star}}{i\hbar} \quad (2.17)$$

where the commutator is computed using the star product (2.16),  $[A, B]_{\star} = A \star B - B \star A$ . Stationary states change simply by a time-dependent phase and are defined via the ‘star-genvalue equation’

$$H \star W_{\psi} = E W_{\psi} , \quad (2.18)$$

which is the time-independent Schrödinger equation determining the energy eigenvalues  $E$ . This approach to quantization can successfully capture the quantum mechanics of the standard textbook examples, but it also has various conceptual subtleties and limitations; for example, Wigner functions in general need not be positive and so do not genuinely determine probability distributions.

### 2.3. Nonassociative quantum mechanics

Following [29] we can apply the phase space formulation to develop a version of nonassociative quantum mechanics that rather remarkably passes all basic physical consistency tests, and moreover has great quantitative power to compute new physical consequences of nonassociativity. Developing nonassociative quantum mechanics also brings into question many foundational issues and can teach us a lot about the nature of quantum theory itself. We shall work throughout with the case where  $\rho := \frac{e}{c} \nabla \cdot \mathbf{B}$  is a constant monopole density, and choose the gauge

$$\mathbf{B}(\mathbf{x}) = \frac{\rho c}{3e} \mathbf{x} . \quad (2.19)$$

The first ingredient we need is a suitable modification of the Moyal star product (2.16). Such a nonassociative star product was first constructed in [7]. Here we shall use the form developed in [29] (see also [30]), and hence introduce the nonassociative phase space ‘monopole star product’ via a simple modification of the Bopp shifts in (2.16) to ‘ $\rho$ -twisted Bopp shifts’ as

$$A(\mathbf{x}, \mathbf{p}) \star B(\mathbf{x}, \mathbf{p}) = A\left(\mathbf{x} - \frac{i\hbar}{2} \nabla_{\mathbf{p}}, \mathbf{p} + \frac{i\hbar}{2} (\nabla_{\mathbf{x}} + \frac{1}{3} \rho \mathbf{x} \times \nabla_{\mathbf{p}})\right) B(\mathbf{x}, \mathbf{p}) . \quad (2.20)$$

That this provides a suitable quantization of the magnetic monopole algebra (2.4) and (2.9) can be seen by computing the corresponding star commutators of phase space coordinate functions

$$[x^i, x^j]_{\star} = 0 \quad , \quad [x^i, p_j]_{\star} = i\hbar \delta^i_j \quad , \quad [p_i, p_j]_{\star} = \frac{i\hbar}{3} \rho \varepsilon_{ijk} x^k \quad (2.21)$$

and the corresponding non-vanishing star Jacobiator

$$[p_1, p_2, p_3]_{\star} = \hbar^2 \rho . \quad (2.22)$$



Globally, the monopole star product reproduces the appropriate algebra (2.6) and (2.10) of magnetic translation operators as

$$U(\mathbf{a}_1) \star U(\mathbf{a}_2) = e^{\frac{i}{\hbar} \Phi_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2)} U(\mathbf{a}_1 + \mathbf{a}_2) \quad , \quad \Phi_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2) = \frac{1}{6} \rho(\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{x} \quad (2.23)$$

and

$$\begin{aligned} (U(\mathbf{a}_1) \star U(\mathbf{a}_2)) \star U(\mathbf{a}_3) &= e^{\frac{i}{\hbar} \Phi_3(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)} U(\mathbf{a}_1) \star (U(\mathbf{a}_2) \star U(\mathbf{a}_3)) \quad , \\ \Phi_3(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) &= \frac{\hbar^3}{6} \rho \mathbf{a}_1 \cdot (\mathbf{a}_1 \times \mathbf{a}_2) \quad . \end{aligned} \quad (2.24)$$

The formula (2.20) is exact when  $\rho$  is constant.

The monopole star product has various noteworthy properties which are important for calculations in nonassociative quantum mechanics. First of all, since the star product  $A \star B$  differs from the ordinary pointwise product of functions  $AB$  by total derivative terms, it is ‘2-cyclic’ in the sense that

$$\int d\mathbf{x} d\mathbf{p} A \star B = \int d\mathbf{x} d\mathbf{p} B \star A = \int d\mathbf{x} d\mathbf{p} AB \quad , \quad (2.25)$$

for suitable functions  $A$  and  $B$  of Schwartz class. Hence noncommutativity at this order is washed away upon integration, i.e. “on-shell”. Similarly, since the triple star product  $A \star (B \star C)$  differs from  $(A \star B) \star C$  by total derivative terms, it is ‘3-cyclic’ in the sense that

$$\int d\mathbf{x} d\mathbf{p} A \star (B \star C) = \int d\mathbf{x} d\mathbf{p} (A \star B) \star C \quad . \quad (2.26)$$

Hence nonassociativity at this order is also absent on-shell. However, this is not true for higher order multiple star products in general, see [29] for a general analysis of this feature. Finally, the monopole star product is Hermitian in the sense that it mimicks the usual conjugation properties of operator products,

$$(A \star B)^* = B^* \star A^* \quad , \quad (2.27)$$

and it is unital in the sense that the constant function 1 serves as an identity element for the star product algebra of functions,

$$A \star 1 = A = 1 \star A \quad . \quad (2.28)$$

A state in nonassociative quantum mechanics is specified by a collection of  $L^2$ -normalized phase space wavefunctions  $\psi_a(\mathbf{x}, \mathbf{p})$ ,

$$\int d\mathbf{x} d\mathbf{p} |\psi_a(\mathbf{x}, \mathbf{p})|^2 = 1 \quad , \quad (2.29)$$

together with statistical probabilities  $\mu_a \in [0, 1]$ ,

$$\sum_a \mu_a = 1 \quad . \quad (2.30)$$

Expectation values are then defined by

$$\langle A \rangle_\psi = \sum_a \mu_a \int d\mathbf{x} d\mathbf{p} \psi_a^* \star (A \star \psi_a) \quad , \quad (2.31)$$

which using 2-cyclicity and 3-cyclicity of the monopole star product can be written as

$$\langle A \rangle_\psi = \int d\mathbf{x} d\mathbf{p} W_\psi(\mathbf{x}, \mathbf{p}) A(\mathbf{x}, \mathbf{p}) \quad (2.32)$$

where

$$W_\psi = \sum_a \mu_a \psi_a \star \psi_a^* \quad , \quad \int d\mathbf{x} d\mathbf{p} W_\psi(\mathbf{x}, \mathbf{p}) = 1 \quad (2.33)$$

plays the role of a Wigner distribution function and should be thought of here as a ‘density operator’.

Now let us demonstrate that some of the basic physical requirements of quantum theory are satisfied by this setup. Let us first check reality of measurements. With the definition (2.31), using Hermiticity and 3-cyclicity we have

$$\langle A \rangle_\psi^* = \sum_a \mu_a \int d\mathbf{x} d\mathbf{p} (A \star \psi_a)^* \star \psi_a = \sum_a \mu_a \int d\mathbf{x} d\mathbf{p} \psi_a^* \star (A^* \star \psi_a) = \langle A^* \rangle_\psi \quad (2.34)$$

and hence observables  $A = A^*$  have real expectation values; thus physical measurements are real in this quantum theory. A similar calculation establishes positivity of measurements [29].

Next we show that observables  $A = A^*$  have real eigenvalues. Using Hermiticity the conjugate of the star-genvalue equation  $A \star W_\psi = \lambda W_\psi$  is  $W_\psi^* \star A = \lambda^* W_\psi^*$  which gives

$$W_\psi^* \star (A \star W_\psi) - (W_\psi^* \star A) \star W_\psi = (\lambda - \lambda^*) (W_\psi^* \star W_\psi) . \quad (2.35)$$

In the associative case we would be done at this stage, because the left-hand side of this equation would vanish, while the right-hand would be generically non-zero, but this is not so in the nonassociative case. However, the left-hand side vanishes on-shell by 3-cyclicity, so by integrating both sides we get

$$0 = (\lambda - \lambda^*) \int d\mathbf{x} d\mathbf{p} W_\psi^* \star W_\psi = (\lambda - \lambda^*) \int d\mathbf{x} d\mathbf{p} |W_\psi(\mathbf{x}, \mathbf{p})|^2 \quad (2.36)$$

and now the right-hand side is non-zero unless  $\lambda = \lambda^*$ . These calculations illustrate the general features of checks in nonassociative quantum mechanics: In all cases the calculations proceed as in the associative case, but there are always a few extra steps required due to nonassociativity. But against all odds, the properties (2.25)–(2.28) of the monopole star product ensure that this nonassociative deformation of quantum theory passes all consistency checks. Various other basic features, including quantum uncertainty relations, are worked out in [29].

The key feature that makes such a nonassociative deformation tractable is the occurrence of a 3-cocycle controlling associativity, as mentioned before; such an algebra should be more properly referred to as ‘quasi-associative’, since it would be hopeless to try to make things work in an arbitrary nonassociative algebra. In the present case it means that there is a multiplicative associator  $\Phi$  which controls the rebracketing of triples of phase space functions under the monopole star product (2.20) as [10, 29]

$$A \star (B \star C) \xrightarrow{\Phi_{A,B,C}} (A \star B) \star C , \quad (2.37)$$

which is most easily described by passing to Fourier space and inserting the 3-cocycle phase factors

$$\Phi_{k,k',k''} = e^{-\frac{i\hbar^2}{6} \rho \mathbf{k}_P \cdot (\mathbf{k}'_P \times \mathbf{k}''_P)} , \quad (2.38)$$

depending only on the Fourier wavenumbers  $\mathbf{k}_p$  dual to momentum coordinates  $\mathbf{p}$ , into the Fourier transformations of triple star products. The associators satisfy ‘pentagon relations’ which are captured by the commuting diagram

$$\begin{array}{ccc}
 & (A \star B) \star (C \star D) & \\
 \Phi_{A \star B, C, D} \swarrow & & \searrow \Phi_{A, B, C \star D} \\
 ((A \star B) \star C) \star D & & A \star (B \star (C \star D)) \\
 \Phi_{A, B, C} \otimes 1 \uparrow & & \downarrow 1 \otimes \Phi_{B, C, D} \\
 (A \star (B \star C)) \star D & \xleftarrow{\Phi_{A, B \star C, D}} & A \star ((B \star C) \star D)
 \end{array} \tag{2.39}$$

and Mac Lane’s coherence theorem asserts that this uniquely defines the insertion of suitable associator factors into higher order iterated star products. This principle enables the extension of our considerations here to the construction of some nonassociative quantum field theories [31,32].

#### 2.4. Momentum space quantization

Let us now examine some of the surprising physical consequences of nonassociativity. One of the standard results in ordinary quantum mechanics is the statement that a pair of observables which do not commute with each other cannot be simultaneously diagonalised. Here we find a higher version of this statement [29]: Nonassociating observables cannot have common eigenstates. In particular, this applies to the diagonalisation of the basic coordinate operators  $x^I \star W_\psi = \lambda^I W_\psi$ , with  $x^I \in (\mathbf{x}, \mathbf{p})$ . From (2.22), we find that the components of momentum  $\mathbf{p}$  cannot be simultaneously measured, which implies a coarse-graining of the momentum space with a uniform monopole background.

We can quantify this quantisation more precisely by defining oriented area uncertainty operators

$$\mathcal{A}^{IJ} = \text{Im}([\tilde{x}^I, \tilde{x}^J]_\star) = -i(\tilde{x}^I \star \tilde{x}^J - \tilde{x}^J \star \tilde{x}^I) \tag{2.40}$$

which mimic the classical formula for the area of a triangle in terms of the vector product between coordinate vectors in directions  $I$  and  $J$ , where we defined the shifted coordinates  $\tilde{x}^I := x^I - \langle x^I \rangle_\psi$  appropriate to the computation of quantum uncertainties. Similarly, we define oriented volume uncertainty operators

$$\mathcal{V}^{JK} = \frac{1}{3} \text{Re}(\tilde{x}^I \star [\tilde{x}^J, \tilde{x}^K]_\star + \tilde{x}^K \star [\tilde{x}^I, \tilde{x}^J]_\star + \tilde{x}^J \star [\tilde{x}^K, \tilde{x}^I]_\star) \tag{2.41}$$

which mimic the classical formula for volume of a tetrahedron in terms of the triple scalar product of coordinate vectors in directions  $I$ ,  $J$  and  $K$ .

One can straightforwardly compute the expectation values of these operators to obtain the non-vanishing minimal areas [29]

$$\langle \mathcal{A}^{x^i, p_j} \rangle_\psi = \hbar \delta^i_j \quad , \quad \langle \mathcal{A}^{p_i, p_j} \rangle_\psi = \frac{\hbar}{3} \rho \varepsilon_{ijk} \langle x^k \rangle_\psi . \tag{2.42}$$

The first equation simply displays the standard Planck cells in phase space of area  $\hbar$  arising from the Heisenberg uncertainty principle of quantum mechanics. The second equation is new, and is due to the uncertainty between momentum measurements proportional to the position measurement in the direction perpendicular to the plane of the momenta, which itself is subject to the standard Planck cell uncertainty. Similarly, one obtains the non-vanishing minimal volume [29]

$$\langle \mathcal{V}^{p_1, p_2, p_3} \rangle_\psi = \frac{1}{2} \hbar^2 \rho \quad (2.43)$$

which clearly indicates a quantized momentum space with a quantum of minimal volume  $\frac{1}{2} \hbar^2 \rho$ . This forbids, in particular, configurations with definite localised momentum in the monopole background.

### 2.5. Testable quantum effects

Let us now take a brief interlude to consider the question of whether it is possible to see experimental signatures of nonassociativity in quantum mechanics. Following [33], we add a confining force with harmonic oscillator potential

$$V(\mathbf{x}) = \frac{1}{2} m \omega^2 \mathbf{x}^2, \quad (2.44)$$

and compute the effective potential due to quantum corrections by

$$V_{\text{eff}} := \langle H \rangle_\psi \big|_{\langle p_i \rangle_\psi = 0}. \quad (2.45)$$

In the present case, the expectation values of the Hamiltonian (2.2) can be computed by using the  $\rho$ -twisted Bopp shifts in (2.20) to write its action on a phase space wavefunction  $\psi(\mathbf{x}, \mathbf{p})$  as

$$H(\mathbf{x}, \mathbf{p}) \star \psi(\mathbf{x}, \mathbf{p}) = \hat{H} \psi(\mathbf{x}, \mathbf{p}), \quad (2.46)$$

where  $\hat{H}$  is the second order differential operator

$$\begin{aligned} \hat{H} = & \frac{1}{2m} \left( \mathbf{p}^2 + i \hbar \mathbf{p} \cdot \nabla_{\mathbf{x}} - \frac{\hbar^2}{4} \nabla_{\mathbf{x}}^2 + \frac{i \hbar}{3} \rho \mathbf{p} \cdot (\mathbf{x} \times \nabla_{\mathbf{p}}) - \frac{\hbar^2}{6} \rho \mathbf{x} \cdot (\nabla_{\mathbf{x}} \times \nabla_{\mathbf{p}}) \right. \\ & \left. - \frac{\hbar^2}{36} \rho^2 ((\mathbf{x} \cdot \nabla_{\mathbf{p}})^2 - \mathbf{x}^2 \nabla_{\mathbf{p}}^2) \right) + \frac{1}{2} \omega^2 \left( \mathbf{x}^2 - i \hbar \mathbf{x} \cdot \nabla_{\mathbf{p}} - \frac{\hbar^2}{4} \nabla_{\mathbf{p}}^2 \right). \end{aligned} \quad (2.47)$$

One can then express the quantum corrections in terms of the fluctuations  $(\Delta x^I)^2 := \langle \tilde{x}^I \star \tilde{x}^I \rangle_\psi$  to get

$$V_{\text{eff}} = V(\langle \mathbf{x} \rangle_\psi) + \frac{1}{2m} (\Delta \mathbf{p})^2 + \frac{1}{2} m \omega^2 (\Delta \mathbf{x})^2. \quad (2.48)$$

Following the standard prescription known from the associative case, the uncertainty moments can be obtained by solving Ehrenfest-type equations of motion derived from the time evolution equations

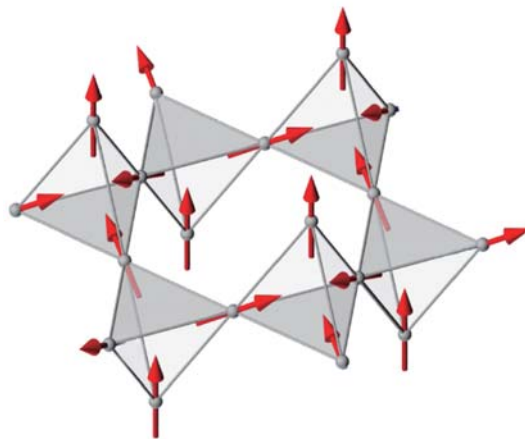
$$\frac{d\langle A \rangle_\psi}{dt} = \frac{\langle [A, H]_\star \rangle_\psi}{i \hbar} \quad (2.49)$$

in an adiabatic approximation, and in the semi-classical limit one obtains [33]

$$V_{\text{eff}} = V(\langle \mathbf{x} \rangle_\psi) + \frac{1}{6m} \hbar \rho |\langle \mathbf{x} \rangle_\psi| + \frac{1}{2} \hbar \omega. \quad (2.50)$$

Hence in addition to the usual zero point energy shift of the quantum harmonic oscillator, the effective potential demonstrates that the motion of electrons in a magnetic monopole density exhibits anharmonic deviations from the classical harmonic motion.

This effect could have potentially observable consequences in certain analogue systems of magnetic monopoles in condensed matter physics, see e.g. [34–36]. These experiments use rare earth oxide insulators  $R_2M_2O_7$  where R is a magnetic ion (such as dysprosium or holmium) and M is a non-magnetic ion (such as titanium). The rare earth atoms R sit at the vertices of two intertwining pyrochlore lattices formed by corner-sharing tetrahedra, as illustrated in Figure 1. Electron spin provides a magnetic dipole at each vertex atom, which is shared by two regular tetrahedra, giving the lattice the geometry of quantum spin ice. The ground state contains two inward and two outward pointing dipoles towards the center of each tetrahedron, and hence has no magnetic charge. Flipping a magnetic dipole at one vertex produces a local topological excitation in which one of the two tetrahedra meeting there has an extra dipole pointing inwards while the other has an extra dipole pointing outwards, giving one tetrahedron three north poles and one south pole, and its neighbouring tetrahedron three south poles and one north pole.



**Figure 1.** Spin ice pyrochlore lattice with magnetic dipoles (taken from [37]).

Iterating this process thus gives an interpretation of flipped magnetic dipoles in terms of tetrahedra containing free and unconfined magnetic monopoles, leading as before to translation group 3-cocycles which are given by the magnetic charge inside a tetrahedron. Scattering neutrons off the material reorganises the magnetic dipoles into a “spin-spaghetti” network of Dirac strings connecting pairs of monopoles, through which magnetic flux flows. These can be observed with applied magnetic fields at low temperature through interference effects by their interaction with the neutrons, which themselves carry a magnetic dipole moment. In such experimental scenarios, magnetic monopoles therefore exist as emergent states of matter. While these configurations do not involve a constant monopole density as in our theoretical framework, it is conceivable that the dynamics of electrons moving in a spin ice pyrochlore lattice could be approximated by electrons moving in a uniform magnetic charge distribution on scales larger than the lattice spacing, whereas the effects of Dirac strings may survive on averaging to a continuous distribution.

### 3. Jordanian quantum mechanics and the octonions

We shall now return to some general algebraic considerations and compare the magnetic monopole algebra with the nonassociative algebras that arose in the early days of quantum mechanics. This will lead us into discussing the algebra of octonions, which we shall show is related to the monopole algebra via a contraction. By suitably interpreting the former algebra in this context, we will thereby arrive at our nonassociative deformation of three-dimensional quantum gravity.

#### 3.1. Noncommutative Jordan algebras

The idea of nonassociativity in quantum mechanics is not new, and traces back to the beginning days of quantum theory through the work of Jordan [38], who attempted to assemble quantum observables into an algebraic framework. The basic observation of Jordan is the following. If  $A$  and  $B$  are Hermitian operators, then their product  $AB$  is not a Hermitian operator unless they commute, i.e.  $AB$  is not observable unless  $A$  and  $B$  can be simultaneously measured; thus Hermitian operators do not close to an algebra under the usual operator product. And neither is their commutator  $[A, B]$  an observable. However, the symmetrised product

$$A \circ B := \frac{1}{2} (AB + BA) \quad (3.1)$$

is a Hermitian operator. The product  $\circ$  is evidently commutative,

$$A \circ B = B \circ A, \quad (3.2)$$

but it is not associative. However, it still satisfies the ‘Jordan identity’

$$(A^2 \circ B) \circ A = A^2 \circ (B \circ A). \quad (3.3)$$

A commutative nonassociative algebra over  $\mathbb{R}$  whose product  $\circ$  satisfies (3.3) for all elements  $A$  and  $B$  is called a (linear) Jordan algebra. If the algebra can be embedded in an associative algebra such that its nonassociative product  $\circ$  is given as in (3.1) then the Jordan algebra is said to be “special”. Jordan’s hope was to find a system which was not derived from an associative algebra in this way, but behaved as one, in order to dispell of the insufficiencies that refer to the underlying unobservable operator algebras.

The theory of Jordan algebras was subsequently developed into a solid mathematical component of algebra, see e.g. [39, 40], wherein it was shown that it was sufficient to replace (3.3) with the weaker condition that the algebra be “alternative”:

$$(A \circ B) \circ A = A \circ (B \circ A), \quad (3.4)$$

and this can be used to define the notion of a noncommutative Jordan algebra. Jordan’s hope of reformulating quantum mechanics in this algebraic framework was dashed by the theorem, proven originally for finite-dimensional algebras in [41] and extended to the infinite-dimensional case in [42], which essentially ruled out the existence of non-trivial Jordan algebras: The only non-special Jordan algebra (up to isomorphism) is the algebra of  $3 \times 3$  Hermitian matrices over the division algebra of octonions  $\mathbb{O}$ , which defines the 27-dimensional “exceptional” Jordan algebra; the octonion algebra will be described in more detail below where it will play a prominent role. This somewhat exotic “octonionic quantum mechanics” satisfies the von Neumann axioms of quantum theory [43], despite the absence of a Hilbert space formulation. For reviews and further details, see e.g. [44, 45].

A natural question at this stage, in view of our previous version of nonassociative quantum mechanics, is the following: Is the magnetic monopole algebra a noncommutative Jordan

algebra? This question has been answered in the negative, originally by [46], and subsequently via a more systematic treatment in [47] (see also [12]). The simplest counterexample already appears at quadratic order in the phase space coordinates [46]: Whereas one generically has the alternativity relations

$$(x^I \star x^K) \star x^I = x^I \star (x^K \star x^I) \quad , \quad ((x^I)^{\star 2} \star x^K) \star x^I = (x^I)^{\star 2} \star (x^K \star x^I) \quad , \quad (3.5)$$

the nonvanishing associator

$$(\mathbf{p}^2 \star \mathbf{p}^2) \star \mathbf{p}^2 - \mathbf{p}^2 \star (\mathbf{p}^2 \star \mathbf{p}^2) = -\frac{2i}{9} \hbar^2 \rho^2 \mathbf{x} \cdot \mathbf{p} \neq 0 \quad (3.6)$$

demonstrates that the monopole star product does not define an alternative algebra.

This violation of alternativity is of course related to the volume quantization of momentum space that we observed earlier, and it brings into question if all is not lost in this theory; for example, this result would appear to rule out the existence of free stationary states. However, one only needs to ensure that  $\langle \mathbf{x} \cdot \mathbf{p} \rangle_\psi = 0$  in the preparation of states for measurement. An example of such a state in the case of a free particle,  $\rho = V = 0$ , can be constructed from phase space wavefunctions  $\psi(\mathbf{x}, \mathbf{p})$  that solve the Schrödinger equation

$$\frac{1}{2m} \mathbf{p}^2 \star \psi(\mathbf{x}, \mathbf{p}) = \frac{1}{2m} \hat{\mathbf{p}}^2 \psi(\mathbf{x}, \mathbf{p}) = E \psi(\mathbf{x}, \mathbf{p}) \quad , \quad (3.7)$$

where the first order differential operators

$$\hat{p}_i = p_i + \frac{i\hbar}{2} \frac{\partial}{\partial x^i} \quad (3.8)$$

are mutually commuting Hermitian operators, so they have simultaneous eigenfunctions  $\psi_{\mathbf{k}}(\mathbf{x}, \mathbf{p})$  with

$$\hat{p}_i \psi_{\mathbf{k}}(\mathbf{x}, \mathbf{p}) = k_i \psi_{\mathbf{k}}(\mathbf{x}, \mathbf{p}) \quad (3.9)$$

and eigenvalues  $k_i \in \mathbb{R}$  for  $i = 1, 2, 3$ . These three equations are solved by

$$\psi_{\mathbf{k}}(\mathbf{x}, \mathbf{p}) = e^{-\frac{2i}{\hbar} (\mathbf{k} - \mathbf{p}) \cdot \mathbf{x}} \varphi(\mathbf{p}) \quad (3.10)$$

where  $\varphi(\mathbf{p})$  is an arbitrary function of  $\mathbf{p}$  independent of the position coordinates  $\mathbf{x}$ . Substituting into (3.7) then gives the expected energy eigenvalues

$$E = E_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} \quad , \quad (3.11)$$

and this represents the quantization of the free particle in the phase space formulation of quantum mechanics.

Equivalently, the Schrödinger equation may be cast as a star-eigenvalue equation

$$\frac{1}{2m} \mathbf{p}^2 \star W_\psi = E W_\psi \quad (3.12)$$

for the real-valued density operator  $W_\psi(\mathbf{x}, \mathbf{p})$  (see e.g. [28]). This collapses the equation to a pair of partial differential equations, its real and imaginary parts. The imaginary part

$$\mathbf{p} \cdot \nabla_{\mathbf{x}} W_\psi(\mathbf{x}, \mathbf{p}) = 0 \quad (3.13)$$

restricts  $W_\psi(\mathbf{x}, \mathbf{p}) = W_\psi(\mathbf{p})$  to be independent of  $\mathbf{x}$ . The real part

$$\left(\mathbf{p}^2 - \frac{\hbar^2}{4} \nabla_{\mathbf{x}}^2 - 2mE\right) W_\psi(\mathbf{p}) = 0 \quad (3.14)$$

is satisfied for arbitrary real functions  $W_\psi(\mathbf{p})$  of momentum with the energy eigenvalues

$$E = \frac{\mathbf{p}^2}{2m}. \quad (3.15)$$

When a non-vanishing background monopole charge  $\rho$  is turned on, we can consider again separately the real and imaginary parts of the corresponding star-genvalue equation (2.18). From (2.47) with  $\omega = 0$  the imaginary equation can be written as

$$\frac{1}{2m} \mathbf{p} \cdot \hat{\mathbf{D}} W_\psi(\mathbf{x}, \mathbf{p}) = 0 \quad (3.16)$$

where

$$\hat{\mathbf{D}} = \nabla_{\mathbf{x}} - \frac{\rho}{3} \mathbf{x} \times \nabla_{\mathbf{p}}. \quad (3.17)$$

The solutions  $W_\psi(\mathbf{x}, \mathbf{p})$  of this equation are the union of the integral curves of the vector field  $(\frac{1}{m} \mathbf{p}, \frac{\rho}{3m} \mathbf{p} \times \mathbf{x})$  on phase space. The corresponding characteristic equations are

$$m \dot{\mathbf{x}} = \mathbf{p}, \quad m \dot{\mathbf{p}} = \frac{\rho}{3} \mathbf{p} \times \mathbf{x}, \quad (3.18)$$

which are simply the classical equations of motion in the monopole background. One now needs to find from these flow equations a triple of classical integrals of motion  $\mathbf{I} = (I_1, I_2, I_3)$ , such that the general solution is  $W_\psi(\mathbf{x}, \mathbf{p}) = W_\psi(\mathbf{I})$ , but such a complete set does not seem to exist [8] (for the free particle with  $\rho = 0$  we took  $\mathbf{I} = \mathbf{p}$ ).

### 3.2. Octonions and magnetic monopoles

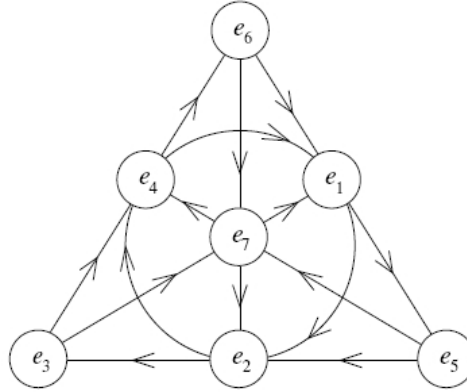
Let us now raise the question of whether there is *any* relation of the magnetic monopole algebra to Jordanian quantum mechanics. This question was answered affirmatively by [11], but to describe that result we first need to take a slight algebraic detour. A foundational result in algebra states that there are only four normed division algebras over the field of real numbers: The real numbers  $\mathbb{R}$  themselves, the complex numbers  $\mathbb{C}$ , the quaternions  $\mathbb{H}$ , and the octonions  $\mathbb{O}$ . They fit into a hierarchy  $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$  where the first two algebras are commutative and associative, the third is noncommutative but associative, while the last one is both noncommutative and nonassociative but is alternative. An element of the octonion algebra  $\mathbb{O}$  is a linear combination of generators

$$a_0 1 + a_1 e_1 + a_2 e_2 + \cdots + a_7 e_7 \quad (3.19)$$

where  $a_i \in \mathbb{R}$ , the element 1 is central and a unit for the algebra, and the seven imaginary unit octonions  $e_i$ , with  $e_i^2 = -1$  and  $e_i e_j = -e_j e_i$  for  $i \neq j$ , have a multiplication rule which can be represented diagrammatically through a mnemonic of the Fano plane, a finite projective plane with seven points and seven lines, as illustrated in Figure 2. Each line contains three points, and each of these triples has a cyclic ordering such that if  $e_i$ ,  $e_j$  and  $e_k$  are cyclically ordered in this way then

$$e_i e_j = e_k = -e_j e_i. \quad (3.20)$$





**Figure 2.** Fano plane mnemonic for multiplication of imaginary unit octonions  $e_1, \dots, e_7$ .

This defines an alternative multiplication which makes  $\mathbb{O}$  into a finite-dimensional noncommutative Jordan algebra; see e.g. [45] for a pedagogical introduction.

For our purposes it is most convenient to relabel the imaginary units as  $e_i, f_i, e_7$  with  $i = 1, 2, 3$  and  $f_1, f_2, f_3 = e_4, e_5, e_6$ , and to represent the multiplication rule by

$$\begin{aligned} e_i e_j &= -\delta_{ij} + \varepsilon_{ijk} e_k, & e_7 e_i &= f_i, \\ f_i f_j &= -\delta_{ij} - \varepsilon_{ijk} e_k, & e_7 f_i &= e_i, \\ e_i f_j &= \delta_{ij} e_7 - \varepsilon_{ijk} f_k. \end{aligned} \quad (3.21)$$

The first relation shows that  $e_i$  for  $i = 1, 2, 3$  generate an associative quaternion subalgebra  $\mathbb{H} \subset \mathbb{O}$ . The non-vanishing Jacobiators are given by

$$\begin{aligned} [e_i, e_j, f_k] &= 4(\varepsilon_{ijk} e_7 + \delta_{kj} f_i - \delta_{ki} f_j), \\ [e_i, f_j, f_k] &= -4(\delta_{ij} e_k - \delta_{ik} e_j), \\ [f_i, f_j, f_k] &= -4\varepsilon_{ijk} e_7, \\ [e_i, e_j, e_7] &= -4\varepsilon_{ijk} f_k, \\ [e_i, f_j, e_7] &= -4\varepsilon_{ijk} e_k, \\ [f_i, f_j, e_7] &= 4\varepsilon_{ijk} f_k. \end{aligned} \quad (3.22)$$

Following [11], let us now rescale the octonionic units and set

$$x^i = -\frac{i\hbar}{2} \lambda e_i, \quad p_i = \frac{i\hbar}{2\sqrt{3}} \sqrt{\lambda\rho} f_i, \quad I = \frac{i\hbar}{2\sqrt{3}} \sqrt{\lambda^3\rho} e_7 \quad (3.23)$$

for a parameter  $\lambda \in \mathbb{R}$ , and take the contraction limit  $\lambda \rightarrow 0$  of the octonionic commutation relations following from (3.21). The various commutators then contract in the following way:

$$\begin{aligned} [e_i, e_j] &= 2\varepsilon_{ijk} e_k \xrightarrow{\lambda \rightarrow 0} [x^i, x^j] = 0, \\ [f_i, f_j] &= -2\varepsilon_{ijk} e_k \xrightarrow{\lambda \rightarrow 0} [p_i, p_j] = \frac{i\hbar}{3} \rho \varepsilon_{ijk} x^k, \\ [e_i, f_j] &= 2(\delta_{ij} e_7 - \varepsilon_{ijk} f_k) \xrightarrow{\lambda \rightarrow 0} [x^i, p_j] = i\hbar \delta^i_j I, \\ [e_7, e_i] &= 2f_i, \quad [e_7, f_i] = -2e_i \xrightarrow{\lambda \rightarrow 0} [x^i, I] = 0 = [p_i, I]. \end{aligned} \quad (3.24)$$

Moreover, only one Jacobiator from (3.22) contracts non-trivially and is given by

$$[f_1, f_2, f_3] = -4e_7 \xrightarrow{\lambda \rightarrow 0} [p_1, p_2, p_3] = \hbar^2 \rho I. \quad (3.25)$$

This demonstrates a surprising relationship between magnetic monopoles and the octonions [11]: The contracted octonion algebra is the magnetic monopole algebra given by (2.4) and (2.9), with central element  $I$ . This result was extended by [12] to a complete quantization of algebras of functions by an explicit construction of an exact (though somewhat complicated) star product which quantizes the octonions, and contracts non-trivially to the monopole star product (2.20).

### 3.3. Nonassociative quantum gravity

By relabelling  $I = p_4$ , the uncontracted octonion algebra reads as

$$\begin{aligned} [x^i, x^j] &= i\hbar \lambda \varepsilon^{ijk} x^k, \\ [p_i, p_j] &= \frac{i\hbar}{3} \rho \varepsilon_{ijk} x^k, \quad [p_4, p_i] = -\frac{i\hbar}{3} \lambda \rho x_i, \\ [x_i, p_j] &= i\hbar \delta_{ij} p_4 - i\hbar \lambda \varepsilon_{ijk} p_k, \quad [x^i, p_4] = -i\hbar \lambda^2 x^i. \end{aligned} \quad (3.26)$$

This algebra describes a seven-dimensional nonassociative phase space with an “extra” momentum mode  $p_4$ ; the non-trivial Jacobiators can be read off by substituting (3.23) into (3.22). Let us now elucidate the physical meaning of this phase space algebra, and in particular the contraction parameter  $\lambda$ , following [13].

Looking at the quaternionic (or  $SU(2)$ ) subalgebra generated by the position coordinates  $x^i$  in (3.26), setting  $\rho = 0$  reveals a noncommutative but associative deformation of spacetime with  $[p_i, p_j] = 0$ . The combination  $\lambda^2 \mathbf{p}^2 + p_4^2$  is straightforwardly calculated to be a central element of this algebra. Setting it to 1, and then restricting the momenta to the upper hemisphere of this sphere to eliminate  $p_4$ , the remaining non-vanishing commutation relations from (3.26) become

$$[x^i, x^j] = i\hbar \lambda \varepsilon^{ijk} x^k, \quad [x_i, p_j] = i\hbar \sqrt{1 - \lambda^2 \mathbf{p}^2} \delta_{ij} - i\hbar \lambda \varepsilon_{ijk} p_k. \quad (3.27)$$

We therefore recover in this way the deformed phase space commutation relations (1.1) of three-dimensional quantum gravity provided we identify

$$\lambda = \frac{\ell_P}{\hbar}. \quad (3.28)$$

This observation suggests a novel nonassociative deformation of three-dimensional quantum gravity, wherein the uncontracted octonion algebra (3.26) appears to be related to monopoles in the spacetime of three-dimensional quantum gravity, with the contraction parameter identified with the Planck length through (3.28). The meaning of the contraction limit  $\lambda \rightarrow 0$  is then clear: It corresponds to turning off quantum gravitational effects, leaving the nonassociative quantum mechanics of electrons in the background of a constant magnetic charge density.

## 4. Epilogue: Magnetic monopoles in M-theory

The interpretation of the nonassociative phase space algebra (3.26) as describing the quantum mechanics of electrons propagating simultaneously in magnetic charge and gravitational backgrounds is very tantalising in light of recent suggestions from non-geometric string theory, which anticipate a nonassociative theory of gravity to govern the low-energy effective dynamics of closed strings in certain locally non-geometric backgrounds [4, 9, 10, 32, 48, 49]. However, the appearance of the three-sphere in momentum space, which is a characteristic feature of

three-dimensional quantum gravity, is put in here by hand as a restriction of a four-dimensional momentum space. This suggests that a deeper structure is inherent in the physics governed by the phase space relations (3.26), and it remains to find a concrete physical realisation.

One possible such scenario was proposed by [13] in the framework of M-theory, which by definition is a quantum theory of gravity in 11 dimensions. The starting point is to embed magnetic monopoles into IIA string theory as D6-branes, where the corresponding electron probes are D0-branes. The D6-brane lifts to M-theory as a Kaluza-Klein monopole described by the 11-dimensional metric

$$ds_{11}^2 = ds_7^2 + H d\mathbf{x} \cdot d\mathbf{x} + H^{-1} (dx^4 + \mathbf{A} \cdot d\mathbf{x})^2, \quad (4.1)$$

where  $x^4 \in S^1$  is the M-theory direction and the harmonic function  $H(\mathbf{x})$  is defined by

$$\nabla \times \mathbf{A} = \nabla H, \quad \nabla^2 H = \rho. \quad (4.2)$$

The corresponding electron probes lifting D0-branes are M-waves along  $S^1$ , which are graviton momentum modes with

$$p_4 = \frac{\hbar e}{R_{11}}. \quad (4.3)$$

Because the probes are waves, they have no local position along the M-theory circle; hence the coordinate  $x^4$  does not appear in the phase space algebra (3.26), which we propose to describe the M-waves in the Kaluza-Klein monopole background.

The relevant parameters in IIA string theory are the string length  $\ell_s$  and the string coupling  $g_s$ , while those of M-theory are the 11-dimensional Planck length  $\ell_P$  and the radius  $R_{11}$  of the M-theory circle. They are related through

$$\ell_s^2 = \frac{\ell_P^3}{R_{11}}, \quad g_s = \left( \frac{R_{11}}{\ell_P} \right)^{3/2}. \quad (4.4)$$

The reduction of M-theory to IIA string theory is defined by the limit  $g_s, R_{11} \rightarrow 0$  with  $\ell_s$  finite. With the identification (3.28), this implies the contraction limit  $\lambda \sim \ell_P \sim R_{11}^{1/3} \rightarrow 0$  reducing the M-wave algebra (3.26) to the magnetic monopole algebra (2.4) of the D0-branes. In this limit the graviton momentum mode (4.3) is frozen and hence disappears from the phase space, while the Kaluza-Klein quantum number  $e$  becomes the electric charge of the D0-branes.

For a single Dirac monopole, wherein  $\rho$  is a delta-function distribution, the four-dimensional part of (4.1) is the metric of the hyper-Kähler Taub-NUT space with local coordinates  $(\mathbf{x}, x^4) \in \mathbb{R}^3 \times S^1$ . For the uniform distribution of magnetic charge that we are interested in here,  $\rho$  can be written as an integral over infinitely-many densely distributed Dirac monopoles. In this case, there is no local expression for the vector potential  $\mathbf{A}$  and hence no local expression for the metric (4.1); the resulting solution can thereby be thought of as a ‘non-geometric Kaluza-Klein monopole’. Similar to what occurred before, the emergence of local non-geometry here arises because the local  $S^1$ -fibration over  $\mathbb{R}^3$  described by the Taub-NUT space is replaced by an  $S^1$ -gerbe over  $\mathbb{R}^3$ . For further details, see [13].

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## References

- [1] 't Hooft G 1996 *Class. Quant. Grav.* **13** 1023–1040 (*Preprint gr-qc/9601014*)
- [2] Matschull H J and Welling M 1998 *Class. Quant. Grav.* **15** 2981–3030 (*Preprint gr-qc/9708054*)
- [3] Freidel L and Livine E R 2006 *Phys. Rev. Lett.* **96** 221301 (*Preprint hep-th/0512113*)
- [4] Blumenhagen R and Plauschinn E 2011 *J. Phys. A* **44** 015401 (*Preprint 1010.1263*)
- [5] Lüst D 2010 *J. High Energy Phys.* **12** 084 (*Preprint 1010.1361*)
- [6] Blumenhagen R, Deser A, Lüst D, Plauschinn E and Rennecke F 2011 *J. Phys. A* **44** 385401 (*Preprint 1106.0316*)
- [7] Mylonas D, Schupp P and Szabo R J 2012 *J. High Energy Phys.* **09** 012 (*Preprint 1207.0926*)
- [8] Bakas I and Lüst D 2014 *J. High Energy Phys.* **01** 171 (*Preprint 1309.3172*)
- [9] Blumenhagen R, Fuchs M, Hassler F, Lüst D and Sun R 2014 *J. High Energy Phys.* **04** 141 (*Preprint 1312.0719*)
- [10] Aschieri P and Szabo R J 2015 *J. Phys. Conf. Ser.* **634** 012004 (*Preprint 1504.03915*)
- [11] Günaydin M, Lüst D and Malek E 2016 *J. High Energy Phys.* **11** 027 (*Preprint 1607.06474*)
- [12] Kupriyanov V G and Szabo R J 2017 *J. High Energy Phys.* **02** 099 (*Preprint 1701.02574*)
- [13] Lüst D, Malek E and Szabo R J 2017 Non-geometric Kaluza-Klein monopoles and magnetic duals of M-theory *R*-flux backgrounds (*Preprint 1705.09639*)
- [14] Zak J 1964 *Phys. Rev.* **134** A1602–A1606
- [15] Dirac P A M 1948 *Phys. Rev.* **74** 817–830
- [16] Jackiw R 1985 *Phys. Rev. Lett.* **54** 159–162
- [17] Wu T T and Yang C N 1976 *Nucl. Phys. B* **107** 365–380
- [18] Barnes G E, Schenkel A and Szabo R J 2015 *J. Geom. Phys.* **89** 111–152 (*Preprint 1409.6331*)
- [19] Bunk S, Saemann C and Szabo R J 2016 The 2-Hilbert space of a prequantum bundle gerbe (*Preprint 1608.08455*)
- [20] Bunk S and Szabo R J 2017 *Lett. Math. Phys.* **107** 1877–1918 (*Preprint 1612.01878*)
- [21] Günaydin M and Zumino B 1985 *Old and New Problems in Fundamental Physics: Meeting in Honour of G.C. Wick* pp 43–53
- [22] Günaydin M and Minic D 2013 *Fortsch. Phys.* **61** 873–892 (*Preprint 1304.0410*)
- [23] Bojowald M, Brahma S, Büyükçam U and Strobl T 2015 *J. High Energy Phys.* **03** 093 (*Preprint 1411.3710*)
- [24] Grönwold H 1946 *Physica* **12** 405–460
- [25] Moyal J E 1949 *Proc. Cambridge Phil. Soc.* **45** 99–124
- [26] Weyl H 1927 *Z. Phys.* **46** 1–46
- [27] Wigner E 1932 *Phys. Rev.* **40** 749–759
- [28] Zachos C K 2002 *Int. J. Mod. Phys. A* **17** 297–316 (*Preprint hep-th/0110114*)
- [29] Mylonas D, Schupp P and Szabo R J 2014 *J. Math. Phys.* **55** 122301 (*Preprint 1312.1621*)
- [30] Kupriyanov V G and Vassilevich D V 2015 *J. High Energy Phys.* **09** 103 (*Preprint 1506.02329*)
- [31] Mylonas D and Szabo R J 2014 *Fortsch. Phys.* **62** 727–732 (*Preprint 1404.7304*)
- [32] Barnes G E, Schenkel A and Szabo R J 2016 *Proc. Science CORFU2015* 081 (*Preprint 1601.07353*)
- [33] Bojowald M, Brahma S and Büyükçam U 2015 *Phys. Rev. Lett.* **115** 220402 (*Preprint 1510.07559*)
- [34] Castelnovo C, Moessner R and Sondhi S L 2008 *Nature* **451** 42–45 (*Preprint 0710.5515*)
- [35] Morris D J P *et al.* 2009 *Science* **326** 411–414 (*Preprint 1011.1174*)
- [36] Ray M W, Ruokokoski E, Tiurev K, Möttönen and Hall D S 2015 *Science* **348** 544–547
- [37] Machida Y, Nakatsuji S, Onoda S, Tayama T and Sakakibara T 2010 *Nature* **463** 210–213
- [38] Jordan P 1932 *Nachr. Ges. Wiss. Göttingen* **1932** 569–575
- [39] Albert A A 1946 *Trans. Amer. Math. Soc.* **59** 524–555
- [40] Schafer R D 1955 *Bull. Amer. Math. Soc.* **61** 469–484
- [41] Jordan P, von Neumann J and Wigner E 1934 *Ann. Math.* **35** 29–64
- [42] Zelmanov E I 1983 *Sibirsk Mat. J.* **24** 89–104
- [43] Günaydin M, Piron C and Ruegg H 1978 *Commun. Math. Phys.* **61** 69–85
- [44] McCrimmon K 1978 *Bull. Amer. Math. Soc.* **84** 612–627
- [45] Baez J C 2012 *Found. Phys.* **42** 819–855 (*Preprint 1101.5690*)
- [46] Held A and Schupp P 2014 unpublished
- [47] Bojowald M, Brahma S, Büyükçam U and Strobl T 2017 *J. High Energy Phys.* **04** 028 (*Preprint 1610.08359*)
- [48] Barnes G E, Schenkel A and Szabo R J 2016 *J. Geom. Phys.* **106** 234–255 (*Preprint 1507.02792*)
- [49] Blumenhagen R and Fuchs M 2016 *J. High Energy Phys.* **07** 019 (*Preprint 1604.03253*)